DYNAMICS OF A RECTILINEAR ISOLATED CRACK GROWING AT A CONSTANT RATE UNDER CONDITIONS OF ANTIPLANE DEFORMATION

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Article [1] discusses the problem of the development of a rectilinear crack under the conditions of antiplane deformation with its arbitrary loading and an arbitrary law of the motion of its ends. The difficulty in the practical application of the solution obtained consists in the need for a consecutive calculation of all the wave diffractions running along the crack from one of its edges to the other. Under these circumstances, for large times (in comparison with the time of the passage of a wave over the length of the crack) it is practically impossible to find a solution. With this aspect, the best properties are those of self-similar solutions, obtained in the problem of the development of a rectilinear isolated crack from zero at a constant rate under the action of the corresponding load. Here the arbitrary law of the loading can be approximated [2] by the sum of self-similar loads. For the case of plane and axisymmetric deformation, several such self-similar problems have been solved [3-9]. In the present article, analogs of these problems for antiplane deformation are considered as a partial case. The consideration of antiplane deformation is explained, on the one hand, by the great mathematical simplicity of this case in comparison with plane deformation and, on the other hand, by the fact that many of the qualitative aspects of the solutions in the cases of plane and antiplane deformation are common.

1. Self-similar Problems

As is well known [1], with antiplane deformation, the sole component of the vector of the displacements differing fron zero is w, the displacement along the z axis. The function $w(x; y, t)$ satisfies the wave equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2 w}{\partial t^2},$$

where b is the velocity of the transverse waves. The components of the stress tensor, differing from zero, are expressed in terms of w in the following manner:

$$\tau_{yz} = \mu \partial w/\partial y,\ \tau_{xz} = \mu \partial w/\partial x, \tag{1.1}$$

where μ is the shear modulus.

In what follows we shall seek solutions of individual problems, both for w and for its derivatives with respect to the time $v_n = \partial^{(-n)} w/\partial t^{(-n)}$, $n = -1, -2, \ldots$, and the first transformations of v_n, such that $w = \partial^n v_n/\partial t^n$, $n = 0, 1, 2, \ldots$. All the functions $v_n(x, y, t)$ also satisfy the wave equation.

Let us consider a class of self-similar solutions, consisting of solutions of the wave equation for v_n, representing homogeneous functions of the variables x, y, t of zero order. Such solutions are completely described by a class of functionally invariant Smirnov–Sobolev solutions [10]. Thus, a given function $v_n(x/bt, y/bt)$, satisfying the wave equation, can be represented in the form

$$v_n(x/bt,\ y/bt) = \operatorname{Re} V_n(z), \tag{1.2}$$

where $V_n(z)$ is an analytical function of the complex variable z, connected with the variables x, y, t by the relationship

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$$\delta \equiv t - zx - y\sqrt{b^{-2} - z^2} = 0.$$

We give here formulas for replacement of the variables with differentiation:

$$\frac{\partial}{\partial t} = -\frac{1}{\delta'}\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial x} = \frac{z}{\delta'}\frac{\partial}{\partial z}, \; \frac{\partial}{\partial y} = \frac{\sqrt{b^{-2} - z^2}}{\delta'}\frac{\partial}{\partial z}, \quad \delta' = \frac{\partial\delta}{\partial z}. \tag{1.3}$$

In particular, $z = t/x$, $\delta' = -x$ with $y = 0$.

Relationship (1.2) allows us to reduce the problem for a wave equation to the problem of finding the analytical function $V_n(z)$. Here the initial and boundary conditions of the problem for the wave equation must go over into the boundary conditions for finding $V_n(z)$.

Using the connection between $v_n(x/bt, y/bt)$ and the displacement $w(x, t, t)$ and formulas (1.1), the result can be obtained that the components of the stress tensor have the form

$$\tau_{yz} = -\tau_0 (t_0/t)^{n-1} \tau^0_{yz} (x/bt, y/bt).$$

In accordance with this, we consider the problem of the expansion of a crack from zero at a constant rate v; the sides of the crack are loaded by the shear stress

$$\tau(x, t) = \pm\tau_0(t_0/t)^{n-1} f(x/bt), \tag{1.4}$$

where $f(s)$ is an arbitrary integrable function; x is the coordinate along the line of the crack.

The simplest solution is obtained with $n = 0$. In this case, the solution reduces to finding the function $V_0(z_1)$, $z_1 = 1/bz$, whose real part is the displacement

$$w(x/bt, y/bt) = \operatorname{Re} V_0(z_1).$$

As is well known [10], the upper half-plane of the plane $z_1 = x_1 + iy_1$ corresponds in the physical plane to the interior of the semicircle $x^2 + y^2 \leqslant b^2t^2$. Here the segment of the axis $y_1 = 0$, $|x_1| < v/b$, corresponds to the cut $|x| < vt$; $v/b < |x_1| < 1$, to the sections of the x axis from the tip of the cut to the wave; $|x_1| > 1$ corresponds to the wave $x^2 + y^2 = b^2t^2$, $y > 0$. An infinitely removed point in the plane z_1 corresponds to the physical point of the wave $x = 0$, $y = bt$.

Let us write the boundary conditions of the problem under discussion at the x_1 axis, the plane z_1. With $y_1 = 0$, $|x_1| < v/b$, there is given $\tau_{yz} = -\tau_0 t_0 f(x_1)/t$. Using (1.1)-(1.3) and the condition $y = 0$ in this interval, we obtain

$$\operatorname{Re} V_0' \sqrt{b^{-2} - z^2} = \operatorname{Re} M_0 = \tau_0 t_0 b x_1 f(x_1)/\mu.$$

Here and in what follows a prime denotes differentiation with respect to the complex variable z. With $y_1 = 0$, $v/b < |x_1| < 1$, the displacements $w(x/bt, 0)$ are equal to zero, i.e., $\operatorname{Re} V_0(x_1) = 0$. With $y_1 = 0$, $|x_1| > 1$, $w = 0$ as a consequence of the zero initial data of the problem, from which it follows that, in this section also, $\operatorname{Re} V_0(x_1) = 0$.

For the function $M_0(z_1) = V_0'\sqrt{b^{-2} - z^2}$ we have the following boundary conditions with $y_1 = 0$:

$$\begin{aligned} &\operatorname{Re} M_0 = \tau_0 t_0 x_1 b f(x_1)/\mu && \text{with} \quad |x_1| < vt, \\ &\operatorname{Im} M_0 = 0 && \text{with} \quad v/b < |x_1| < 1, \\ &\operatorname{Re} M_0 = 0 && \text{with} \quad |x_1| > 1. \end{aligned} \tag{1.5}$$

For an unambiguous determination of the analytical function $M_0(z_1)$ from the boundary conditions (1.5), the character of the behavior of the sought function must be determined at the points $z_1 = 0, \infty, \pm v/b, \pm 1$, starting from the physical statement of the problem.

1. The point $z_1 = 0$, $z = \infty$, corresponds to the middle of the crack. We postulate the boundedness of the displacement at this point $V_0 \sim a_0 + a_1/z + \ldots$ (a_0, a_1, ... are arbitrary real numbers). From this $V_0' \sim 1/z^2$, $M_0 \sim i/z \sim iz_1$.

2. The point $z_1 = \infty$, $z = 0$ corresponds to the point of the wave $x = 0$, $y = bt$. Since $\mathrm{Re}\, V_0(z_1) = 0$ on both sides of the point $z = 0$ with $\mathrm{Re}\, z = 0$, for reasons of symmetry $V_0(z_1)$ can be prolonged continuously beyond the real axis to the right and left of $z = 0$. This means that the point $z = 0$ cannot be a branch point of the function $V_0(z_1)$. The Laurent expansion at this point has the form

$$V_0(z_1) \sim i(a_0 + a_1 z + \ldots).$$

The absence of terms with negative powers is explained, on the one hand, by the fact that there is no reason to expect a special behavior of the solution at this point of the wave in comparison with other points of the wave, and, on the other hand, by the fact that the appearance of such terms would mean the loading of this point by some external action. It can be shown that the function $V_0 = 2Q/i\pi\mu bz$ is a solution of the problem of the elastic field of the antiplane displacements, arising in a plane when it is loaded by pairs of concentrated forces $\pm Q$, running along the y axis with the velocity b. As a result, with $z_1 \to \infty$, we have $M_0(z_1) \sim i$.

3. With $z_1 = \pm v/b$, which corresponds to the tips of the crack, we shall assume, with $M_0(z_1)$, the presence of a root singularity $(z_1 - v^2/b^2)^{-1/2}$, since, in the present problem, the value of $\mathrm{Re}\, M_0(z_1)$ is proportional to τ_{yz}.

4. The points $z_1 = \pm 1$ correspond to points of the wave at the x axis. Analogously to the case $z_1 = \infty$, it can be shown that these points cannot be branch points for $V_0(z_1)$, while the presence of the terms $(z - b^{-})^{-m}$, $m > 0$, in the Laurent expansion with $z \to b^{-1}$ would mean the additional concentrated loading of this point. From this, at this point, we postulate the following form of the expansion: $V_0(z_1) \sim i[a_0 + a_1(z - b^{-1}) + \ldots]$, from which it follows that, with $z_1 \to \pm 1$, $M_0 \sim \sqrt{z_1^2 - 1}$. The function $M_0(z_1)$, satisfying the boundary conditions (1.5) and having the required behavior at the points $z_1 = 0, \infty, \pm v/b, \pm 1$, is singular:

$$M_0(z_1) = \frac{z_1\sqrt{z_1^2 - 1}}{\pi i \sqrt{z_1^2 - v^2/b^2}} \int_{-v/b}^{v/b} \frac{t_0\tau_0 b f(s)}{\mu(s - z_1)} \sqrt{\frac{s^2 - v^2/b^2}{s^2 - 1}}\, ds.$$

We give an expression for the coefficient of the intensity of the stresses. If the stresses near the tip of the crack have the asymptotic $\tau_{yz} \approx N/\sqrt{x - vt}$, then

$$N = \frac{\tau_0 t_0 b \sqrt{1 - v^2/b^2}}{\pi\sqrt{2vt}} \int_{-v/b}^{v/b} \sqrt{\frac{v/b + x}{v/b - x}} \frac{f(x)\, dx}{\sqrt{1 - x^2}}. \tag{1.6}$$

From this, with $f(x) = 1$, which corresponds to a constant loading along the crack $\tau(x, t) = \tau_0 t_0/t$, it follows that

$$N = 2N^0 K(v/b) \sqrt{1 - v^2/b^2}/\pi,$$

where $N^0 = \tau_0 t_0 (v/2t)^{1/2}$ is the value of the coefficient of the intensity for the static problem of the uniform loading of a crack with a length 2vt by a load $\tau_0 t_0/t$; $K(v/b)$ is a total elliptical integral of the first kind.

With $f(x) = Q\delta_1(x)/\tau_0 b t_0$, which corresponds to loading by a constant force, from (1.6) it follows that

$$N = N^0 \sqrt{1 - v^2/b^2}, \quad N^0 = Q/\pi\sqrt{2vt}, \tag{1.7}$$

where N^0 is the value of the coefficient of the intensity for the static problem of a crack with a length of 2vt, loaded by shear forces ±Q. The components of the stress tensor τ_{yz} in this case in the interval vt < x < bt are determined by the expression

$$\tau_{yz} = Qv\sqrt{b^2t^2 - x^2}/\pi bx\sqrt{x^2 - v^2t^2}. \tag{1.8}$$

The case n = 1 in (1.4) is investigated analogously to the preceding [11]. Here the coefficient of the intensity is determined by the expression

$$N = \frac{\tau_0 t_0^2\sqrt{vt/2}}{\pi t^2\sqrt{1 - v^2/b^2}}\frac{b}{v}\int\limits_{-v/b}^{v/b}\frac{f(s)\sqrt{s - v/b}\sqrt{1 - s^2}}{\sqrt{v/b - 1}}\,ds. \tag{1.9}$$

From (1.9), for a uniformly distributed load, follows

$$\tau(x, t) = \tau_0(t_0/t)^2\delta_0(1 - |x/vt|),$$
$$N = 2N^0E(v/b)/\pi\sqrt{1 - v^2/b^2},\ N^0 = \tau_0(t_0/t)^2\sqrt{vt/2};$$

for a concentrated load

$$\tau(x, t) = Qt_0\delta_1(x)/t,$$
$$N = N^0/\sqrt{1 - v^2/b^2},\ N^0 = Qt_0/\pi t\sqrt{2vt}.$$

In the latter case, τ_{yz}, in the interval between the tip of the crack and the wave at the axis y = 0, is determined by the expression

$$\tau_{yz} = Qt_0v/\pi b^2t^2x_1\sqrt{x_1^2 - v^2/b^2}\sqrt{1 - x_1^2}. \tag{1.10}$$

For a value of $n_1 \geqslant 2$, it is not possible to construct a solution satisfying the boundary conditions and the required behavior at the points $Z_1 = 0, \infty, \pm v/b, \pm 1$, determined analogously to what was done for n = 0, 1.

To understand the reason for this, let us find the dipole moment D of the loading forces for the partial case of loading by a constant stress $\tau = \tau_0(t_0/t)^{n+1}$ along the length of the crack. Such a dipole moment is proportional to τv^2t^2. Thus, for n = 0, D ∿ t. For n = 1, D ∿ const, which corresponds to the problem of the inclusion of the dipole at the origin of coordinates at the zero moment of time. In the case n = 2, D ∿ t^{-1}. Such a loading leads to infinite displacements, analogously to loading by the half-plane force Q(t) (the Lamb problem), not integrable at the initial moment of time.

Let us now investigated the case $n \leqslant -1$ (n = k − 1; k = 0, −1, −2, ...). With such a loading law

$$v_n(x/bt, y/bt) = \mathrm{Re}\,V_n(z_1) = \partial^{1-k}w/\partial t^{(1-k)},$$
$$\partial^{(1-k)}\tau_{yz}/\partial t^{(1-k)} = \mu\,\mathrm{Re}\,V_n'\sqrt{b^{-2} - z^2}/\delta' = \mu\,\mathrm{Re}\,M_n/\delta'.$$

From this, with $|x_1| < v/b$, $y_1 = 0$; when the external load is given (1.4), we have Re $M_n = (x/\mu)\partial^{(1-k)}\tau(x, t)/\partial t^{(1-k)}$. For $v/b < |x_1| < 1$, w = 0, $\partial^{(1-k)}w/\partial t^{(1-k)} = 0$, Re $V_n = 0$, Im $M_n = 0$. With $|x_1| > 1$, Re $V_n = 0$, Re $M_n = 0$.

Let us evaluate the character of the behavior of the functions $M_n(z_1)$ at the points z = 0, ±v/b, ±1, ∞, with the conditions $f(x_1) = f(-x_1)$ in (1.4):

for $z_1 = 0$, $V_n \sim a_0 + a_2z^2 + \ldots$, $M_n \sim iz_1^2$;

$$\text{for } z_1 = \infty, \; V_n \sim i(a_1 z + a_3 z^3 + \ldots), \; M_n \sim i;$$

$$\text{for } z_1 = \pm v/b, \; M_n \sim \left(z_1^2 - v^2/b^2\right)^{-(3-2k)/2};$$

$$\text{for } z_1 = \pm 1, \; V_n \sim i\,[a_0 + a_1(z \mp b^{-1}) + \ldots], \quad M_n \sim \sqrt{z_1^2 - 1}.$$

The function satisfying all these requirements has the form

$$M_n = \frac{z_1 \sqrt{z_1^2 - 1}}{\left(z_1^2 - v^2/b^2\right)^{(3-2k)/2}} \left[\frac{1}{\pi i} \int\limits_{-v/b}^{v/b} \frac{\operatorname{Re} M_n \,(s^2 - v^2/b^2)^{(3-2k)/2} ds}{s\sqrt{s^2 - 1}\,(s - z_1)} + i z_1 \sum_{m=0}^{|k|} A_m z_1^{2m} \right]. \tag{1.11}$$

where A_m ($m = 0, 1, \ldots, |k|$) are undetermined constants, which must be found by establishing the values of $\partial^p \tau_{yz}/\partial t^n$ ($p = 0, \ldots, |k|$) with respect to M_n at the sides of the crack, and requiring the satisfaction of the corresponding boundary condition at some point of the crack, for example, $x = vt - 0$, $y = 0$. Under these circumstances, a linear system of equations is obtained with respect to A_m:

$$\frac{\partial^p \tau_{yz}}{\partial t^p} = -\frac{\partial^p \tau(vt, t)}{\partial t^p} = \mu \frac{x^{(-k-p)}}{(-k-p)!} \operatorname{Re} \int\limits_{b^{-1}}^{v^{-1}-0} (v^{-1} - t)^{-p-k} M_n dt, \tag{1.12}$$

whose solution completes the construction of the function $M_n(z_1)$ (1.11).

We give expressions for the coefficients of the intensity with uniformly distributed and concentrated stresses, found by the procedure described above with $k = 0$:

$$\tau = \tau_0 \delta_0(1 - |x/vt|),$$
$$N = N^0 \sqrt{1 - v^2/b^2}/E(\sqrt{1 - v^2/b^2}), \quad N^0 = \tau_0 \sqrt{vt/2}; \tag{1.13}$$

$$\tau = Q\delta_1(x) t/t_0,$$
$$N = N^0 \sqrt{1 - v^2/b^2}\,[1 - v^2 K(\sqrt{1 - v^2/b^2})/b^2 E(\sqrt{1 - v^2/b^2})], \; N^0 = Qt/\pi t_0 \sqrt{2vt}. \tag{1.14}$$

An analog of the latter problem for plane deformation is discussed in [7], but, there, in the solution a term is omitted containing an undetermined constant and having the sense of a solution with loading of the sides of the crack by a constant stress (the Broberg problem). Thus, the solution found in [7] corresponds to a combination of a concentrated and some uniform stress.

With an increase in the value of $|k|$, it becomes difficult to obtain analytical dependences similar to (1.13), (1.14), and it obviously will be more realistic to use a numerical method for obtaining and solving the system (1.12) for A_m. From the solutions of the investigated self-similar problems of the theory of the elasticity of the dynamic loading of a crack growing with a rate v it follows that, with $v/b \ll 1$, the coefficient of the intensity of the stress is close to its "quasistatistic" value $N^\circ(t)$, determined from a solution of the static problem. The correction factor to the quasistatistic value of N/N° in the cases under discussion depends essentially on the law of the loading (1.4) and can be either less or greater than unity.

2. Concentrated Loading

The following problem is considered. From the origin of coordinates $x = y = 0$, in an elastic plane at rest at the initial moment of time, a crack $|x| < vt$, $y = 0$ starts to develop itself to the right and left at a constant rate v. The sides of the crack are loaded by a concentrated force,

$$\tau(x, t) = \pm f(t)\delta_1(x)\delta_0(t), \tag{2.1}$$

acting along the z axis. Here $\tau(x, t)$ is the stress of the loading; $\delta_1(x)$ is a Dirac delta-function; $\delta_0(t)$ is a Heaviside function; $f(t)$ is a function, determining the change in the

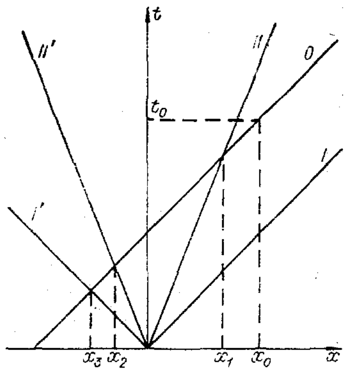

Fig. 1

forces with time; the sign + and — relate to the upper and lower sides of the crack. It is required to determine the stress-intensity coefficient at the ends of the crack.

In [2] it is proposed to approximate an arbitrary loading law of the type (2.1) by a power series with respect to t:

$$f(t) = \sum_{i=0}^{\infty} a_i t^i.$$

This approach makes it possible to bring the solution of the problem posed down to the solution of self-similar problems of the loading of a similar crack by the load

$$\tau(x, t) = \pm t^i \delta_1(x)\delta_0(t), \ i = 0, 1, 2, \ldots . \quad (2.2)$$

Under these circumstances, the solution for the problem (2.1) is obtained as a linear combination of solutions with the loading (2.2), and the stress-intensity coefficient at the crack tip is found in the form of a series

$$N(t) = \sum_{i=0}^{\infty} a_i N_i(t), \quad (2.3)$$

where $N_i(t)$ is the stress-intensity coefficient in a self-similar problem with the load (2.2).

A problem with the load (2.2) is a partial case of the self-similar problems discussed in Sec. 1, which gives analytical expressions for N_i with i = 0 (1.7) and i = 1 (1.14). With i > 1, we find the values of N_i numerically solving the self-similar problem with a load (2.2) by reducing it to an integral Fredholm equation with respect to an unknown component of the stress tensor τ_{yz} with y = 0. For comparison of this equation with respect to τ_{yz} (x, 0, t) with $vt < |x| < bt$ we use the result of [1].

The stress at the point (x_0, 0) at the continuation of a semiinfinite moving and arbitrarily loaded crack $x < l(t)$, y = 0 in an originally quiescent plane at the moment of time t_0 is determined by the expression

$$\tau_{yz}(x_0, t_0) = \frac{1}{\pi\sqrt{x_0 - x_1}} \int_{x_0 - t_0}^{x_1} p(x, t_0 - x_0 + x) \frac{\sqrt{x_1 - x}}{x_0 - x} dx, \quad (2.4)$$

where x_1 is the coordinate of the point of intersection of the characteristic passing through the point (x_0, t_0) and the trajectory of the end of the crack in the plane (x, t); p(x, t) is the loading of the sides of the crack, given with $x < l(t)$.

In the plane (x, t), the problem under consideration has the configuration shown in Fig. 1. The characteristics I and I' separate the regions of rest from the region of motion. The trajectories of the ends of the crack are illustrated by the straight lines II and II'. Between them there is a region where the load p(x,t) is given by formula (2.2).

TABLE 1

i	$v=0,2$	$v=0,4$	$v=0,6$	$v=0,8$
0	0,225 403	0,163 340	0,105 573	$0,513\,167\cdot10^{-1}$
1	0,142 952	$0,630\,221\cdot10^{-1}$	$0,223\,547\cdot10^{-1}$	$0,453\,999\cdot10^{-2}$
2	$0,902\,05\cdot10^{-1}$	$0,241\,791\cdot10^{-1}$	$0,480\,846\cdot10^{-2}$	$0,424\,334\cdot10^{-3}$
3	$0,568\,226\cdot10^{-1}$	$0,937\,153\cdot10^{-2}$	$0,106\,459\cdot10^{-2}$	$0,414\,002\cdot10^{-4}$
4	$0,358\,242\cdot10^{-1}$	$0,367\,91\cdot10^{-2}$	$0,241\,062\cdot10^{-3}$	$0,414\,889\cdot10^{-5}$
5	$0,226\,381\cdot10^{-1}$	$0,146\,107\cdot10^{-2}$	$0,554\,745\cdot10^{-4}$	$0,423\,211\cdot10^{-6}$
6	$0,143\,487\cdot10^{-1}$	$0,585\,757\cdot10^{-3}$	$0,129\,177\cdot10^{-4}$	$0,437\,142\cdot10^{-7}$
7	$0,912\,4\cdot10^{-2}$	$0,236\,64\cdot10^{-3}$	$0,303\,47\cdot10^{-5}$	$0,455\,760\cdot10^{-8}$
8	$0,582\,0\cdot10^{-2}$	$0,961\,9\cdot10^{-4}$	$0,717\,90\cdot10^{-6}$	$0,478\,61\cdot10^{-9}$
9	$0,372\,4\cdot10^{-2}$	$0,393\,0\cdot10^{-4}$	$0,170\,77\cdot10^{-6}$	$0,505\,49\cdot10^{-10}$

Assuming the stresses with $x < vt$ to be given, from (2.4) we obtain an integral equation for τ_{yz}. Here the integration in (2.4) is carried out along the characteristic 0 (see Fig. 1) in the intervals (x_2, x_1), where the load $p(x, t) = \tau(x, t)$ is given from (2.2) and $(x_3 x_2)$, where the load $p(x, t) = -\tau_{yz}(x, t)$ is unknown.

As a consequence of the self-similarity of the problem with the load (2.2), the stresses τ and τ_{yz} are represented in the form

$$\tau = t^{i-1}\delta_1(x/t),\ \tau_{yz} = t^{i-1}\varphi_i(x/t). \tag{2.5}$$

Substituting (2.5) into Eq. (2.4) and going over to the variable $\xi = x/t$, we obtain an integral equation with respect to $\varphi_i(\xi)$:

$$\frac{\varphi_i(\xi_0)}{(1-\xi_0)^i}\left(\frac{\xi_0}{1-\xi_0}-\frac{v}{1-v}\right)^{1/2} = -\frac{1}{\pi}\int_{-1}^{-v}\frac{\varphi_i(\xi)}{(1-\xi)^i}\left(\frac{v}{1-v}-\frac{\xi}{1-\xi}\right)^{1/2}d\xi + \frac{1}{\pi\xi_0}\sqrt{\frac{v}{1-v}}.$$

Using the symmetry of the problem $\varphi_i(\xi) = \varphi_i(-\xi)$ and introducing the new unknown function

$$u_i(\xi) = \varphi_i(\xi)(1-\xi)^{-i-1/2}\pi\sqrt{\xi - v}/\sqrt{v}, \tag{2.6}$$

for $u_i(\xi)$ we obtain the equation

$$u_i(\xi_0) = -\frac{1}{\pi}\int_v^1\frac{u_i(\xi_0)}{\xi_0+\xi}\left(\frac{1-\xi}{1+\xi}\right)^{i+1/2}\left(\frac{\xi+v}{\xi-v}\right)^{1/2}d\xi + \frac{1}{\xi_0}. \tag{2.7}$$

With $i \geqslant -1$, this equation belongs to the class of Fredholm equations.

For $i = -1, 0$, the solution of (2.7) is the function

$$u_i = \sqrt{v}(1+\xi)^{i+1/2}/\pi\xi\sqrt{\xi+v}.$$

These solutions coincide with (1.10), (1.8).

For $i \geqslant 1$, Eq. (2.7) was solved numerically, after bringing it to the following form by a replacement of the variable of the integration, so chosen as to get rid of the root singularities,

$$u_i(x) = -(1-v)^{i+1}\int_0^1\frac{u(s)}{x+s}\cdot\frac{[\cos(\pi t/2)]^{2i+2}\sqrt{v+s}}{(1+s)^{i+1/2}}dt + \frac{1}{x},$$
$$s = v + (1-v)\sin^2(\pi t/2). \tag{2.8}$$

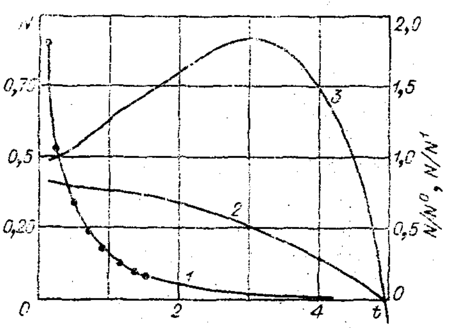

Fig. 2

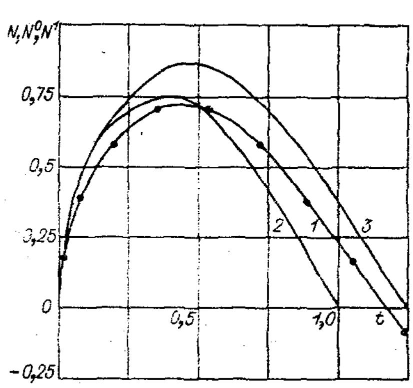

Fig. 3

The integral in (2.8) is represented in the form of a quadrature formula of the Gauss type, with fixed extreme points [12]:

$$\int_{-1}^{1} f(x)\,dx \approx Af(-1) + Bf(1) + \sum_{k=1}^{r} c_k f(x_k),$$

where the order of r is taken equal to 15. The system of linear algebraic equations, obtained from (2.8) for the values of u_i at the points of the quadrature formula x_k, was solved using a standard program.

A comparison between the numerical solution obtained in this manner and an analytical solution with $i = 0$, $v = 0.4$ showed that the relative error in this case does not exceed 10^{-7}.

For large values of i and small values of $(1 - v)$, from Eq. (2.8) it follows that

$$u_i(x) \approx 1/x. \tag{2.9}$$

This was confirmed by a numerical calculation. Thus, already with $v = 0.4$, $i = 1$, the maximal deviation from the asymptotic (2.9) was 1.5% and with a rise in i it decreased very rapidly. To increase the accuracy of the calculations with large values of i, an equation for the function $w_i(x) = 1/x - u_i(x)$, easily obtained from (2.8), was computed numerically. The accuracy was controlled by comparison with a solution found with the use of a quadrature formula of the 10-th order.

Table 1 gives values of $vw_i(v)$, calculated for the case $v = 0.2, 0.4, 0.6, 0.8$ with $0 \leqslant i \leqslant 9$.

Using the value of $vw_i(v)$, we can find the coefficient of the intensity of the stresses with a singularity at the tip of the crack in the problem (2.2). From (2.5), (2.6), it follows that

$$N_i = t^{i-1/2}(1 - v)^{i+1/2}[1 - vw_i(v)]/\pi\sqrt{v}. \tag{2.10}$$

With large values of i and v, from (2.9), (2.10) it follows that

$$N_i \approx t^{i-1/2}(1 - v)^{i+1/2}/\pi\sqrt{v}.$$

We now construct the solution of the problem with the load (2.1), whose function $f(t)$, in the interval $0 < t < \infty$, can be represented by a Taylor series $f(t) = \sum_{i=0}^{\infty} a_i t^i$. In accordance with (2.3), (2.10), the stress-intensity coefficient with such a stress of the crack sides is expressed by the formula

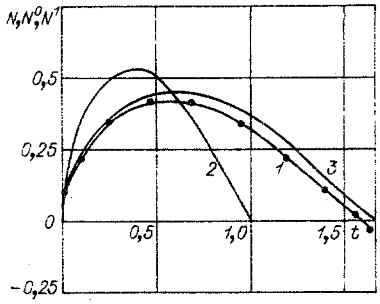

Fig. 4

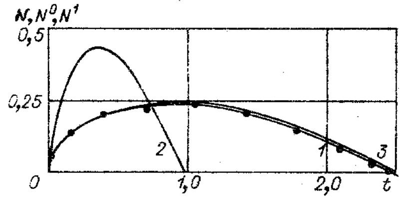

Fig. 5

$$N(t) = \sqrt{1-v}\left[f(s) - v\sum_{i=0}^{\infty} a_i w_i(v) s^i\right] \Big/ \pi\sqrt{vt}, \tag{2.11}$$
$$s = t(1-v),$$

which, with $v \to 1$, has an asymptotic form

$$N(t) \approx N^1(t) = \sqrt{1-v} f[t(1-v)]\,\pi\sqrt{vt}. \tag{2.12}$$

We note that the convergence of the series in expression (2.11) is better than that of the series for $f(t)$, since the factor $w_i(v)$ decreases rapidly with a rise in i.

This makes it possible, with a given cut-off of the series for $f(t)$, using formula (2.11), to obtain N(t) with a value of t greater than is permissible from the condition of the approximation of $f(t)$ by the given cut-off of the series.

As a first example, let us consider the law of change of the forces of the stress with time:

$$f(t) = e^{-t} \approx \sum_{i=0}^{m} (-t)^i / i!. \tag{2.13}$$

The error with calculation of N using formula (2.11) with $t = 16.7$, $v = 0.4$, $m = 16$, is 10^{-4}, while for the series (2.13) with $m = 15$ an error of the same order of magnitude is attained with $t = 3.4$.

Figure 2 shows dependences of N, N/N^0, N/N^1 (curves 1-3, respectively) for the case $v = 0.4$ with $m = 16$. For $N^0(t)$, there is taken here the value of the coefficient of the intensity which is obtained with solution of the problem of the loading of a crack with a length 2vt by the forces $f(t)$:

$$N^0 = f(t)\,\pi\sqrt{2vt}. \tag{2.14}$$

The dependence $N^1(t)$ is determined by formula (2.12).

Qualitatively, the behavior of N(t) with $t > 4.5$ in this case can be described in the following manner: a transition through 0 with $t = 4.96$, a decrease to -10^{-3} with $t = 6.67$, and then a slow rise to $-2 \cdot 10^{-4}$ with $t = 16.7$.

The form of the dependences of N/N^0 and N/N^1 shows that for $v = 0.4$, N(t) differs strongly from both N^0 and N^1. The analogous dependences for $v = 0.2$ and $v = 0.8$ showed that with small velocities N(t) can be well (10%) approximated by $N^0(t)$, and, with high velocities, by $N^1(t)$ in the interval $0 < t < 3$. With $t > 3$, a characteristic feature for N with $v = 0.2$, 0.4, 0.6, 0.8 is a transition through 0 at the moment of time 5.15, 4.95, 7.15, and 20, respectively.

Let us consider the case $f(t) = \sin \pi t$, $0 < t < 1$. This function was approximated by the cutoff of the series $\sin \pi t \approx \sum_{i=0}^{3} (-\pi t)^{2i+1}/(2i+1)!$. The results of calculations using

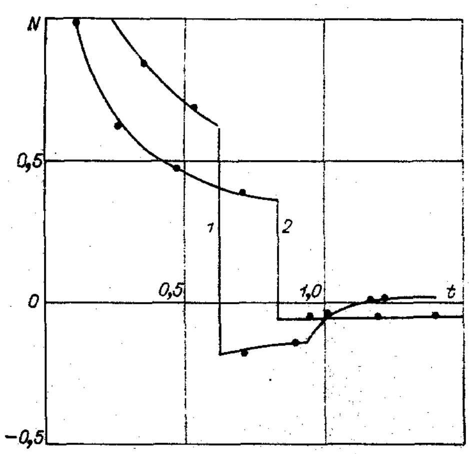

Fig. 6

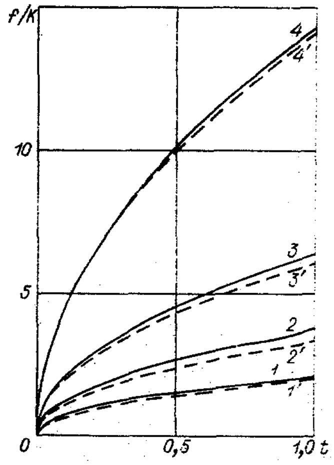

Fig. 7

formula (2.11) with v = 0.2, 0.4, 0.6 are shown on Figs. 3-5, where curves 1-3 correspond to the dependences N(t), $N^0(t)$, and $N^1(t)$; it is evident that, with a decrease in v, the curves of N and N^0 approach each other; with an increase, the curves of N and N^1 approach each other. With middle values of v, neither of these approximations is effective.

We note also that, as in the case of a power law, N(t) reverts to zero, then passes over to negative values.

For calculation of the dependence N(t) in the case of an arbitrary change in the forces $f(t)$, representing, in the interval $0 < t < 1$, a limited function at L_2, we can use (to obtain a polynomial approximation of such a law) an expansion in terms of shifted Legendre polynomials P_n' [13]:

$$f(t) \approx \sum_{n=0}^{r} b_n P_n', \quad b_n = (2n+1)\int_0^1 P_n'(t) f(t)\, dt. \tag{2.15}$$

Representing the integral in (2.15) by a quadrature Gauss formula of order m = r + 1, and using (2.11), we obtain an expression for calculating N(t):

$$N(t_i) \approx \frac{1-v}{\pi\sqrt{vx_i}} \sum_{k=0}^{m} f(x_i)(\delta_{ik} + m_{ik}),$$
$$x_i = (1-v)t_i, \quad i = 1, 2, \ldots, m, \tag{2.16}$$

where x_i are points of the quadrature formula [12]. The matrix m_{ik} is expressed in terms of the weights of the Gauss formula A_k [12], the coefficients P_n^i with the powers t^i of the polynomials P_n' [13], and the values P_{ni} of the polynomials P_n' at the points x_i,

$$m_{ik} = \frac{vA_k}{2} \sum_{s=0}^{r} w_s(v)(-x_i)^s \sum_{n=s}^{r} (2n+1) P_n^s P_{nk}. \tag{2.17}$$

As an example of the use of formulas (2.16), (2.17) (r = 9), calculations were made of the cases considered above, $f(t) = e^{-t}$ and $f(t) = \sin \pi t$. The results are plotted by the points in Figs. 2-5. It can be seen that the agreement is completely satisfactory.

For an evaluation of the accuracy of formulas (2.16), (2.17) (r = 9), we consider a loading law which allows an explicit solution using formula (2.4),

$$f(t) = \delta_0(t)\delta_0(0.5-t), \quad 0 < t < 1.$$

With such a loading, for $0 < t < 0.5/(1-v)$, the values of N and τ_{vz} are determined by expressions (1.7), (1.8). Using them, with $t > 0.5/(1-v)/N(t)$ can be found by numerical integration. Here while in the interval $0.5/(1-v) < t < 0.5(1+v)/(1-v)^2$ we must take account of a single integral, with $0.5(1 + v)/(1 - v)^2 < t < 0.5(1 + v)^2/(1 - v)^3$ we must take account of a double integral, and so forth.

Curves 1, 2 in Fig. 6 show dependences $N(t)$, obtained in this manner with $v = 0.2$ and 0.4. The points plot values of $N(t)$ found for the given loading law using formulas (2.16) and (2.17). It can be seen that the coincidence of the results is completely satisfactory.

Formula (2.16) makes it possible to solve the inverse problem: From a given $N(t)$ find $f(t)$, in particular the following "optimal" problem.

Let the minimal value of N with which the crack can move at a constant rate v without stopping be K = const. With what loading law $f(t)$ will the coefficient of the singularity at the crack tip be equal to K during the whole time of the motion? Inversion of (2.16) allows us to construct such a law. In Fig. 7 the solid curves 1-4 show dependences of $f(t)/K$ with $v = 0.2$, 0.4, 0.6, 0.8 found from the system (2.16) with a constant left-hand part equal to K.

An approximate solution of this problem can be obtained, assuming that $N = N^1(t)$ (2.12). With such an approximation

$$f(t)/K \doteq \pi\sqrt{vt}/(1 - v). \tag{2.18}$$

The dependences (2.18) with $v = 0.4$, 0.6, 0.8 are shown in Fig. 7 by curves 2'-4'.

From a comparison between the exact and approximate solutions it can be seen that, for the given problem, the approximate solution (2.18) is in satisfactory agreement with the exact solution with $v > 0.2$.

With $v < 0.2$, more exact results are given by a quasistatic approximation $N = N^0(t)$ (2.14) (curve 1' for $v = 0.2$).

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